TD3

Exo 3.1

Soit l'EDO : $4xy''(x) - 4y'(x) + x^3 y(x) = 0$ $(\mathcal{E})$

on suppose que y est developpable en série entière : $y(x) = \sum_{n=0}^{\infty} a_n x^n, \forall n \in \mathbb{R}$

On remplace y dans $(\mathcal{E})$ par sa formulation en SE.

$$(\mathcal{E}) \quad \underbrace{\sum_{n=2}^{\infty}}_{\substack{a^2 x(a_0)=0 \\ a^2 x(a_1 x)=0}} 4(n)(n-1) a_n x^{n-1} - \underbrace{\sum_{n=1}^{\infty}}_{a x(a_0)=0} 4n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+3} = 0$$

On change d'indice dans les 3 sommes

$$\sum_{n=2}^{\infty} 4(n)(n-1) a x^{n-1} = \sum_{\tilde{n}+1=2}^{\infty} 4(\tilde{n}+1)(\tilde{n}) a_{\tilde{n}+1} x^{\tilde{n}} \quad (S_1)$$

$\tilde{n} = n-1$ $\quad \to \tilde{n} = 1$

$$\sum_{n=1}^{\infty} 4n a_n x^{n-1} = \sum_{\tilde{n}+1=1} 4(\tilde{n}+1) a_{\tilde{n}+1} x^{\tilde{n}} \quad (S_2)$$

$\to \tilde{n} = 0$

$$\sum_{n=0}^{\infty} a_n x^{n+3} = \sum_{\tilde{n}-3}^{\infty} a(\tilde{n}-3) \cdot x^{\tilde{n}} \quad (S_3)$$

$n+3 = \tilde{n}$ $\quad \tilde{n}-3=0 \Rightarrow \tilde{n}=3$

On regroupe

$$\sum_{\tilde{n}=1}^{\infty} a_{\tilde{n}} = a_1 + a_2 + \sum_{\tilde{n}=3} a_{\tilde{n}}$$

Cela nous donne

(S_2) $4 \times 1\, a_1 + 4 \times 2\, a_2 x + \sum_{n=3}^{\infty} 4(n+1) a_{n+1} x^n + 4 \times 3\, a_3 n^2$

(S_1) $4 \times 2 \times 1\, a_2 x + 4 \times 3 \times 2\, a_3 x^2 + \sum_{n=3}^{\infty} 4(n+1) n\, a_{n+1} x^n$

Ce qui nous donne :

$$-4a_1 + 0a_2 + (24-12) a_3 x^2 + \sum_{n=3}^{\infty} \left[4(n+1)(n-1) a_{n+1} + a_{n-3}\right] x^n$$

Les termes en dehors de la somme nous donne :

$$\begin{cases} a_0 \text{ quelconque} \\ a_1 = 0 \\ a_2 \text{ quelconque} \\ a_3 = 0 \end{cases}$$

La somme nous donne : $4(n+1)(n-1) a_{n+1} + a_{n+3} = 0$

$\Rightarrow a_{n+1} = -\frac{1}{4(n+1)(n-1)} a_{n-3}$

on a $\forall p \in \mathbb{N}$ $\quad a_{4p} = -\frac{1}{4(4p)(4p-2)} a_{4(p-1)}$

Comme $a_1 = 0$ on a $a_{4p+1} = 0 \Leftarrow a_{4p+1} = -\frac{1}{4(4p+1)(4p-1)} a_{4(p-1)+1}$

~~Com~~

$a_{4p+2} = -\frac{1}{4(4p+2)(4p)} a_{4(p-1)+2}$

Comme $a_3 = 0 \Rightarrow a_{4p+3} = 0 \Leftarrow a_{4p+3} = -\frac{1}{4(4p+3)(4p+1)} a_{4(p-1)+3}$

on en déduit déjà que $\boxed{a_{2p+1} = 0 \quad \forall p}$

La relation (1) nous donne :

$$a_{4p} = \frac{(-1)^p}{4^p 2^p 2^p (2p)!}$$

La relation (2) nous donne:

$$a_{4p+2} = \frac{(-1)^p}{4^p 2^p 2^p (2p+1)!}$$

on a donc finalement

$$y(x) = a_0 \sum_{p=0}^{\infty} \frac{(-1)^p}{4^{2p}(2p)!} x^{4p} + a_2 \sum_{p=0}^{\infty} \frac{(-1)^p}{4^{2p}(2p+1)!} x^{4p+2}$$

- $a_0 \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p)!} \left(\frac{x^2}{4}\right)^{2p} = a_0 \cos\left(\frac{x^2}{2}\right)$
- $4 a_2 \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p+1)!} \left(\frac{x^2}{4}\right)^{2p+1} = 4 a_2 \sin\left(\frac{x^2}{4}\right)$

(Exo partiel)

$$\int_{C(0,1)} \frac{dz}{z} = \int_0^{2\pi} \frac{\gamma'(t)\,dt}{\gamma(t)} \Rightarrow \int_0^{2i\pi} \frac{ie^{it}}{e^{it}} dt = 2\pi i$$

$\gamma(t) = \cos(t) + i\sin(t) = e^{it}$

$\forall t \in [0; 2\pi[$

Exo 3.2

Soit f holomorphe

$\rightarrow f(x,y) = P(x,y) + i\,Q(x,y)$

on a CC $\begin{cases} \dfrac{\partial P}{\partial x} = \dfrac{\partial Q}{\partial y} \\ \dfrac{\partial P}{\partial y} = -\dfrac{\partial Q}{\partial x} \end{cases}$

On veut calculer $\Delta P = \dfrac{\partial^2}{\partial x^2}(P) + \dfrac{\partial^2}{\partial y^2}(P)$

On utilise CC : $\dfrac{\partial^2}{\partial x^2}(P) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial x} P\right) \overset{\boxed{CC}}{=} \dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial y} Q\right)$

$\dfrac{\partial^2}{\partial y}(P) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial y} P\right) = \dfrac{\partial}{\partial y}\left(-\dfrac{\partial}{\partial x} Q\right)$

Théorème de Schwarz : si $P_{(x,y)}$ est une fonct° de C^2

Alors : $\dfrac{\partial}{\partial x}\dfrac{\partial}{\partial y} P = \dfrac{\partial}{\partial y}\dfrac{\partial}{\partial x} P$

On a donc : $\Delta P = \dfrac{\partial}{\partial x}\dfrac{\partial}{\partial y} Q - \dfrac{\partial}{\partial y}\dfrac{\partial}{\partial x} Q$

Sachant que $Q \in C^2$ on a donc :

$$\Delta P = \frac{\partial}{\partial x}\frac{\partial}{\partial y} Q - \frac{\partial}{\partial x}\frac{\partial}{\partial y} Q = 0$$

Soit f holomorphe :

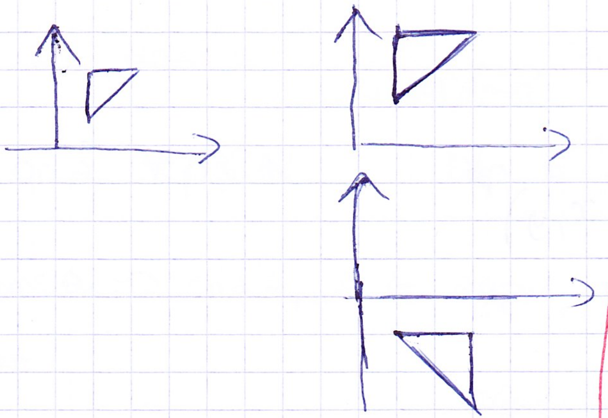
$$J_{f(x,y)} = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix} \overset{CC :}{=} \begin{pmatrix} \frac{\partial Q}{\partial y} & -\frac{\partial Q}{\partial x} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

En posant $a = \dfrac{\partial Q}{\partial y} = \dfrac{\partial P}{\partial x}$ $\quad b = \dfrac{\partial Q}{\partial x} = -\dfrac{\partial P}{\partial y}$

<u>donc $J_{f(x,y)}$ est une similitude directe</u>

En posant $\cos(\theta) = \dfrac{a}{\sqrt{a^2+b^2}}$ et $\sin(\theta) = \dfrac{b}{\sqrt{a^2+b^2}}$

En a : $J_{f(x,y)} = \sqrt{a^2+b^2}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$



$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

3.2.2

$z = x+iy \xrightarrow{f} f(z) = x - iy$

partie réelle $P(x,y) = x$

partie imm $Q(x,y) = -y$

$$\frac{\partial P}{\partial x} = 1 \neq -1 = \frac{\partial Q}{\partial y}$$

$$\underset{\mathbb{R}^2}{\begin{pmatrix} x \\ y \end{pmatrix}} \longrightarrow \underset{\mathbb{R}_2}{\begin{pmatrix} x \\ -y \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \overset{\text{pas relat° CC}}{\Longrightarrow} \text{pas holomorph}$$

3.2.3

Soit $P(x,y) = \ln(x^2+y^2)$

on calcul ΔP : $\Delta P = \frac{\partial^2}{\partial x^2}\left(\ln(x^2+y^2)\right) + \frac{\partial^2}{\partial y^2}\left(\ln(x^2+y^2)\right)$

$$= \frac{\partial}{\partial x}\left(\frac{2x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{2y}{x^2+y^2}\right)$$

$$= \frac{2}{x^2+y^2} + 2x\left(-\frac{x}{(x^2+y^2)^2}\right) + \frac{2}{x^2+y^2} + 2y\left(\frac{2y}{(x^2+y^2)^2}\right)$$

$$= \frac{4}{x^2+y^2} - \frac{4\,(x^2+y^2)}{(x^2+y^2)^2} = 0 \quad \text{donc } P \text{ harmonique}$$

Si P est harmonique alors il existe Q telle que

$$\begin{cases} f(x,y) = P(x,y) + iQ(x,y) \\ f \text{ holomorphe} \end{cases}$$

on cherche Q :

CC $\quad \frac{\partial Q}{\partial y} = \frac{\partial P}{\partial x} = \frac{2x}{x^2+y^2}$

$$Q(x,y) = \int \frac{2x}{x^2+y^2}\,dy + K(x)$$

$$= 2x \int \frac{dy}{\left(1+\frac{y^2}{x^2}\right)x^2} + K(x)$$

$$= 2\int \frac{d(y/x)}{(1+(y/x)^2)} + K(x)$$

$$= 2\operatorname{Arctan}(y/x) + K(x)$$

3.2.3 $\quad$ CC : $\frac{\partial Q}{\partial x} = -\frac{\partial P}{\partial y} = -\frac{2y}{x^2+y^2}$

$\partial/\partial x\,[f(g(x))] = f'(g(x))g'(x)$

On a : $\frac{\partial}{\partial x}\operatorname{Arctan}(y/x) = -\frac{y}{x^2}\,\frac{1}{1+y^2/x^2}$

d'où $\quad \frac{\partial Q}{\partial x} = -2y/x^2\,\frac{1}{1+y^2/x^2} + K'(x) = -\frac{2y}{x^2+y^2} = -\frac{2y}{x^2}\,\frac{1}{1+y^2/x^2}$

On doit avoir égalité des modules :

$$\Rightarrow 1 = |e^{2y}| \Leftrightarrow y = 0$$

donc : $z = x \in \mathbb{R}$

On retrouve le m̂ résultat que dans le cas réel.

$z = \pi/2 + k2\pi \ , \ k \in \mathbb{Z}$

conjugé

- Calcul de $|\cos z|^2 = \cos(z)\,\overline{\cos(z)}$

$$= \left(\frac{e^{iz}+e^{-iz}}{2}\right)\left(\frac{e^{-i\bar{z}}+e^{i\bar{z}}}{2}\right)$$

$$= 1/4\left(e^{i(z-\bar{z})} + e^{i(z+\bar{z})} + e^{-i(z+\bar{z})} + e^{i(\bar{z}-z)}\right)$$

On pose $z = x+iy$

$$\hookrightarrow = 1/4\left(e^{i(iy+iy)} + e^{i(x+x)} + e^{-i(x+x)} + e^{-i(iy+iy)}\right)$$

$$= 1/4\left(e^{-2y} + e^{2ix} + e^{-2ix} + e^{2y}\right)$$

$$= 1/4\left(2\,\mathrm{Ch}(2y) + 2\cos(2x)\right)$$

$$= \frac{1}{2}\,\mathrm{Ch}(2y) + \frac{1}{2}\cos(2x)$$

on a : $\mathrm{Ch}(2y) = 2\mathrm{Ch}^2(y) - 1$

d'où : $|\cos(z)|^2 = \mathrm{Ch}^2(y) + \frac{1}{2}(\cos(2x) - 1)$

sachant que : $\cos(2x) \leq 1, \ \forall x \in \mathbb{R}$

on a donc : $|\cos(z)|^2 \leq \mathrm{Ch}^2(y)$

on a entre autre : $\mathrm{Ch}(2y) = 2(\mathrm{Sh}^2(y) + 1) - 1$

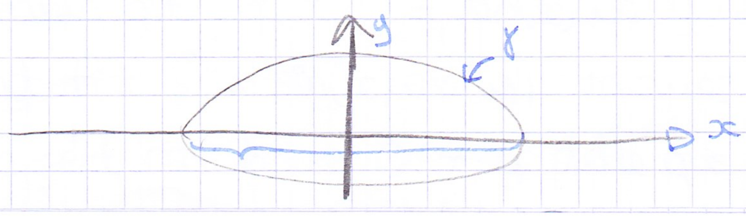
d'où $|\cos(z)|^2 = \mathrm{Sh}^2(y) + \frac{1}{2}(\cos(2x) + 1)$

donc finalement : $|\cos(z)|^2 \geq \mathrm{Sh}(y)^2$

Ex 4.2

$$I = \int_0^{2\pi} \frac{dt}{a^2\cos^2(t) + b^2\sin^2(t)}$$

soit γ telle que : $\gamma(t) = a\cos(t) + ib\cos(t)$



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On retrouve le m̂ résultat que dans le cas réel.

$z = \pi/2 + k2\pi$, $k \in \mathbb{Z}$

- Calcul de $|\cos z|^2 = \cos(z)\,\overline{\cos(z)}$ ← conjugé

$$= \left(\frac{e^{iz}+e^{-iz}}{2}\right)\left(\frac{e^{-i\bar{z}}+e^{i\bar{z}}}{2}\right)$$

$$= 1/4\left(e^{i(z-\bar{z})} + e^{i(z+\bar{z})} + e^{-i(z+\bar{z})} + e^{i(\bar{z}-z)}\right)$$

On pose $z = x+iy$

$$\hookrightarrow = 1/4\left(e^{i(iy+iy)} + e^{i(x+x)} + e^{-i(x+x)} + e^{-i(iy+iy)}\right)$$

$$= 1/4\left(e^{-2y} + e^{2ix} + e^{-2ix} + e^{2y}\right)$$

$$= 1/4\left(2\,\mathrm{Ch}(2y) + 2\cos(2x)\right)$$

$$= \frac{1}{2}\mathrm{Ch}(2y) + \frac{1}{2}\cos(2x)$$

on a : $\mathrm{Ch}(2y) = 2\mathrm{Ch}^2(y) - 1$

d'où : $|\cos(z)|^2 = \mathrm{Ch}^2(y) + \frac{1}{2}(\cos(2x) - 1)$

sachant que : $\cos(2x) \leq 1$, $\forall x \in \mathbb{R}$

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on a entre autre : $\mathrm{Ch}(2y) = 2(\mathrm{Sh}^2(y) + 1) - 1$

d'où $|\cos(z)|^2 = \mathrm{Sh}^2(y) + \frac{1}{2}(\cos(2x) + 1)$

donc finalement : $|\cos(z)|^2 \geq \mathrm{Sh}(y)^2$

Ex 4.2

$$I = \int_0^{2\pi} \frac{dt}{a^2\cos^2(t) + b^2\sin^2(t)}$$

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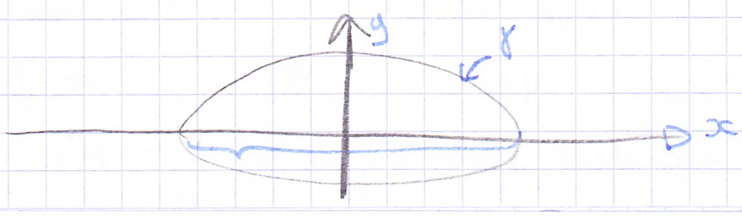

on calcul $K = \int_\gamma \frac{dz}{z} = \int_{C(0,1)} \frac{dz}{z} = 2\pi i$

On a: $\gamma'(t)\overline{\gamma(t)} = (-a\sin(t) + ib\cos(t))(a\cos(t) - ib\sin(t))$

$= -a^2\sin(t)\cos(t) + \underbrace{iab\sin^2(t) + iab\cos^2 t}_{iab} + b^2\cos t\sin t$

$$K = \int_0^{2\pi} \frac{(b^2-a^2)\cos(t)\sin(t)}{a^2\cos^2(t) + b^2\sin^2(t)} dt + \int_0^{2\pi} \frac{iab\ dt}{a^2\cos^2 t + b^2\sin^2 t}$$

$K = 2\pi i$

$$\mathrm{Re}(2\pi i) \overset{=0}{=} \mathrm{Re}(K) = \int_0^{2\pi} \frac{(b^2-a^2)\cos(t)\sin(t)}{a^2\cos^2(t) + b^2\sin^2(t)} dt = 0$$

$$2\pi = \int_0^{2\pi} \frac{ab\ dt}{a^2\cos^2 t + b^2\sin^2(t)} \iff I = \frac{2\pi}{ab}.$$

$|\mathrm{Sh} y| \leq |\cos z| \leq \mathrm{Ch} y$

$\Rightarrow \quad K'(x) = 0$ d'où $K(x) = K$

Finalement on a donc :

$$Q(x,y) = 2\operatorname{Arctan}(y/x) + K$$

3.2.4 $\quad P(x,y) = \sin x \, Ch(y)$

$\Delta P = \cancel{0}$

$= \cancel{\frac{\partial^2}{\partial x^2} P + \frac{\partial^2}{\partial y^2} P}$

$\frac{\partial^2}{\partial x^2} P = -\sin x \, Ch(y)$

$\frac{\partial^2}{\partial y^2} P = \sin(x) \, ch(y)$

donc $\Delta P = 0 \Rightarrow$ harmonique

on cherche $Q(x,y)$ telle que

$$\begin{cases} f(x,y) = P(x,y) + iQ(x,y) \\ f \text{ holomorphe sur } \mathbb{C} \end{cases}$$

$\Rightarrow$ CC $\begin{cases} \partial P/\partial x = \partial Q/\partial y & (1) \\ \partial P/\partial y = -\partial Q/\partial x & (2) \end{cases}$

On utilise les relations de Cauchy

$$Q(x,y) = \int \frac{\partial P}{\partial x} dy + K(x)$$

$$= \int \cos(x) \, Ch(y) \, dy + K(x)$$

$$= \cos(x) \, Sh y + K(x)$$

on utilise (2) $\quad \frac{\partial Q}{\partial x} = -\frac{\partial P}{\partial y}$

$\Leftrightarrow -\sin(x) \, Sh(y) + K'(x) = -\sin(x) \, Sh(y)$

$\Leftrightarrow K'(x) = 0$

$\Leftrightarrow K(x) = K$ (cte indépendante de x)

D'où finalement : $Q(x,y) = \cos(x) \, Sh(y) + K$

on en déduit que $f(x,y) = \sin(x) \, Ch(y) + i\cos(x) \, Sh(y) + iK$

Si on prend $P(x,y) = \sin(x) \cos(y)$ on aura $\Delta f(x,y) = -\sin x \cos x + \sin(x)(-\cos(y))$
$\neq 0, \forall (x,y)$

donc on ne peut pas trouver $Q(x,y)$ telle que $\{$